Spatial Depth-Based Classification for Functional Data

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joint work with
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12th of September 2012, Research workshop: High dimensional and dependent functional data, Bristol, UK.

Outline of the presentation

1. Introduction
2. Functional spatial depths
3. Depth-based supervised functional classification (simulation and real data studies)
4. Conclusions

Functional depth

- There are many reasons why a **standard multivariate data analysis** might fail when data are curves.
- However, the contraposition between MDA and FDA is not total and many multivariate techniques have inspired advances in FDA. An example is represented by the introduction of the **notion of data depth for functional data**.
- The main goal of a functional depth consists in measuring the **degree of centrality** of curves with respect to functional random variables or samples. It can also be useful to build robust FDA methods.
- We present **two new functional depths** which have as starting point a multivariate depth known as **spatial** (Serfling 2002).

Supervised functional classification

- In this work, we tackle the **supervised functional classification problem**.
- We consider three **depth-based procedures** that have been proposed by López-Pintado and Romo (2006) and Cuevas, Febrero and Fraiman (2007). The main goal of their mutual depth-based approach consists in **robust classification**.
- Actually, robustenss might be a key issue in many functional classification problems because the available **FDA outlier detection procedures** are still few.
- **Our main actual research line** consists in the definition of some depth-based outlier detection methods.

The spatial approach

- Let $\mathbf{x} \in \mathbb{R}^d$ and let $S : \mathbb{R}^d \to \mathbb{R}^d$ to be the multivariate spatial sign function given by

$$S(\mathbf{x}) = \begin{cases} \frac{\mathbf{x}}{\|\mathbf{x}\|_E}, & \mathbf{x} \neq \mathbf{0}, \\ \mathbf{0}, & \mathbf{x} = \mathbf{0}, \end{cases}$$

  where $\|\mathbf{x}\|_E$ is the Euclidean norm of $\mathbf{x}$.

- Let $\mathbf{Y}$ to be a random variable with cumulative distribution function F on $\mathbb{R}^d$. Then, the **multivariate spatial depth** of $\mathbf{x}$ with respect to F is defined by

$$SD(\mathbf{x}, F) = 1 - \left\| \int S(\mathbf{x} - \mathbf{y})\, dF(\mathbf{y}) \right\|_E = 1 - \left\| \mathbb{E}\left[S(\mathbf{x} - \mathbf{Y}) \right] \right\|_E ,$$

  (Serfling, 2002).

Functional spatial depth

- Note that $S(\mathbf{x})$ and $SD(\mathbf{x}, F)$ are particular cases of more general definitions for elements and random elements belonging to **normed vector spaces**.
- In this work, we consider these general definitions, but we focus on a different application for each of them: we consider **functional Hilbert spaces**, say $\mathbb{H}$, and we define the **functional spatial sign function** and the **functional spatial depth function** as

$$FS(x) = \begin{cases} \frac{x}{\|x\|}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$FSD(x, P) = 1 - \left\| \mathbb{E}\left[FS(x - Y) \right] \right\|,$$

where now $x \in \mathbb{H}$, Y is a random variable with probability distribution P on $\mathbb{H}$ and $\|\cdot\|$ is the norm defined by the inner product on $\mathbb{H}$.

FSD: interpretation and foundations for a theoretical study

- The uth **functional spatial quantile** of a $\mathbb{H}$-valued random variable Y with probability distribution P is obtained by minimizing w.r.t. q

$$\mathbb{E}\left[\Phi(u, Y-q) - \Phi(u, Y)\right],$$

where $\Phi(u, v) = \|v\| + \langle u, v\rangle$ and $\{u\colon u \in \mathbb{H}, \|u\| < 1\}$. Thus, the uth functional spatial quantile has a **direction** and a **magnitude**.

- Under weak assumptions (Cardot, Cénac and Zitt, 2011), it is possible to define the functional spatial quantile function FQ_P and its inverse FQ_P^{-1}. The latter is given by

$$FQ_P^{-1}(x) = -\mathbb{E}\left[\frac{Y-x}{\|Y-x\|}\right].$$

- Considering norms, we have that

$$\|FQ_P^{-1}(x)\| = \left\|-\mathbb{E}\left[\frac{Y-x}{\|Y-x\|}\right]\right\| = \|\mathbb{E}\left[FS(x-Y)\right]\| = 1 - FSD(x, P).$$

Sample FSD

- When a sample of curves is observed, say Y_n, $FSD(x, P)$ must be replaced to compute the depth value of x with respect to the observed sample. We define the **sample FSD** as

$$FSD_n(x) = 1 - \frac{1}{n} \left\| \sum_{y \in Y_n} FS(x - y) \right\|.$$

- $\|\mathbf{FS}(\mathbf{x} - \mathbf{y})\| = \mathbf{1}$ for any $y \in Y_n$, regardless y is a neighboring or a distant curve from x.
- $FSD_n(x)$ depends on the norm of the sum of n unit-norm curves

  $\Rightarrow$ **Any sample observation contributes equally to $\mathbf{FSD_n(x)}$ $\Rightarrow$ global approach.**
- Can we define a functional depth for which the information brought by each observation depends on its distance from x? Can we implement a **local approach**?

The kernelized functional spatial depth

- Recall that

$$FSD_n(x) = 1 - \frac{1}{n} \left\| \sum_{y \in Y_n} FS(x - y) \right\|.$$

- Note that

$$\left\| \sum_{y \in Y_n} FS(x - y) \right\|^2 = \sum_{y,z \in Y_n} \frac{\langle x, x \rangle + \langle y, z \rangle - \langle x, y \rangle - \langle x, z \rangle}{\sqrt{\langle x, x \rangle + \langle y, y \rangle - 2\langle x, y \rangle}\sqrt{\langle x, x \rangle + \langle z, z \rangle - 2\langle x, z \rangle}}.$$

- The right-hand side of the above equation involves **inner products**, which can also be seen as similarity measures.

The kernelized functional spatial depth

- Based on an idea of Chen et al. (2009), we recode the data to obtain a more powerful similarity measure. We consider a positive definite and stationary **kernel function** instead of the inner product function; that is,

$$\kappa(x, y) = \langle \phi(x), \phi(y) \rangle,$$

where $\phi : x \in \mathbb{H} \to \mathbb{F}$ is an embedding map. ϕ and $\mathbb{F}$ are usually defined implicitly.

- We define the **sample kernelized functional spatial depth** as

$$KFSD_n(x) = 1 - \frac{1}{n} \left(\sum_{y,z \in Y_n} \frac{\kappa(x,x) + \kappa(y,z) - \kappa(x,y) - \kappa(x,z)}{\sqrt{\kappa(x,x) + \kappa(y,y) - 2\kappa(x,y)}\sqrt{\kappa(x,x) + \kappa(z,z) - 2\kappa(x,z)}} \right)^{1/2}.$$

- $KFSD_n(x)$ can be interpreted as a **recoded version** of $FSD_n(x)$.

Choices about the functional space, the kernel function and the bandwidth.

- After introducing FSD and KFSD, we study how they can be useful in functional supervised classification problems.
- For this stage of our research, we do not investigate how the choices about the functional space (and its norm), κ and the kernel bandwidth would affect the behavior of FSD an KFSD. We make three standard choices:

1. **Functional space** (for FSD and KFSD): $\mathbb{H} = L^2[a, b]$.
2. **Kernel function** (for KFSD): $\kappa(x, y) = \exp(-\frac{\|x-y\|^2}{\sigma^2})$.
3. **Kernel bandwidth** (for KFSD): σ = 15th percentile of the empirical distribution of $\{\|y_i - y_j\|, i, j = 1, \ldots, n\}$, as in Febrero, Galeano and González-Manteiga (2008).

Depth-based supervised functional classification methods

- **Theoretical framework for supervised functional classification**: Let $\mathbb{H} = L^2[a,b]$, Y to be a functional random variable and G to be a random variable with value $g = 0$ or $g = 1$. Consider the random pair (Y, G).
- We consider three depth-based methods:

  1. The **distance to the trimmed mean** method (DTM, López-Pintado and Romo 2006).
  2. The **weighted averaged distance** method (WAD, López-Pintado and Romo 2006).
  3. The **within maximum depth** method (WMD, Cuevas, Febrero and Fraiman, 2007).

- We also consider as benchmark a robust method such as the **k-nearest neighbor** procedure (k-NN, Cérou and Guyader 2006, with $k = 5$).

Depth-based supervised functional classification methods

- Assume to observe a sample of $n = n_0 + n_1$ independent pairs, identically distributed as (Y, G), and an independent curve x, identically distributed as Y, but with unknown class membership.

  1. **DTM**: for each of the two groups, DTM computes the α-trimmed mean m_g^α and classifies x in the group for which $\|x - m_g^\alpha\|$ is less. We use $\alpha = 0.2$
  2. **WAD**: for a given group, say $g = 0$, WAD computes a weighted average of the distances $\|x - y_i\|$, where the weights are given by the within-group depth values $D(y_i)$. WAD classifies x in the group for which the weighted averaged distance is less.
  3. **WMD**: for a given group, say $g = 0$, WMD includes the curve x in the sample and computes its depth value, $D(x; g = 0)$. WMD classifies x in the group for which $D(x; \cdot)$ is higher.

Functional depths

- Besides FSD and KFSD, in this work we consider the following functional depths:
  1. The **Fraiman and Muniz depth** (FMD), the first proposed functional depth (Fraiman and Muniz, 2001).
  2. The normalized version of the kernel-based **h-modal depth** (HMD) (Cuevas, Febrero and Fraiman, 2006).
  3. The projection-based **random Tukey depth** (RTD) (Cuesta-Albertos and Nieto-Reyes, 2008).
  4. The projection-based **integrated dual depth** (IDD) (Cuevas and Fraiman, 2009).
  5. The graph-based **modified band depth** (MBD) (López-Pintado and Romo, 2009).

Simulation study: introduction

- Pairing the 3 depth-based methods with the 7 functional depths, we obtain **21 classification procedures + 1 benchmark**.
- For example, with DTM+FSD we refer to the procedure obtained by using the DTM method together with the FSD.
- The **simulation study** is partially based on the ones performed by López-Pintado and Romo (2006) and Cuevas, Febrero and Fraiman (2007), but it contains some slight differences.
- It can be divided in **two parts**: in the first one, we do not allow for contaminated data, whereas in the second one we allow for them through a contamination probabibility given by q.

Simulation study: no contamination

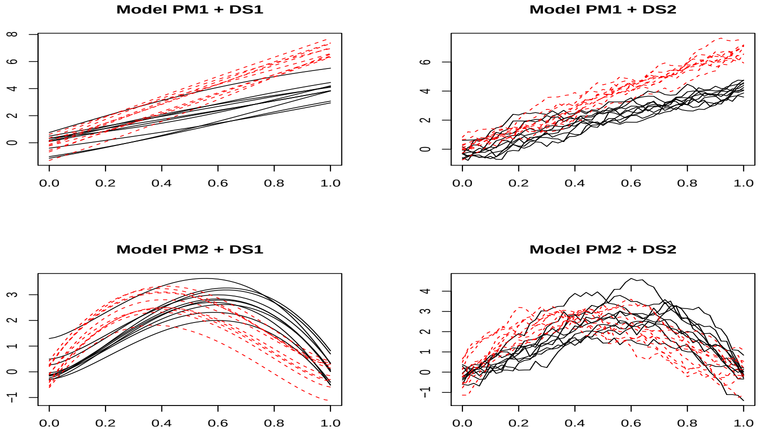
- In absence of contamination, the curves generating processes have the following common structure:

$$x(t) = m_g(t) + \epsilon(t), \quad t \in [0, 1],$$

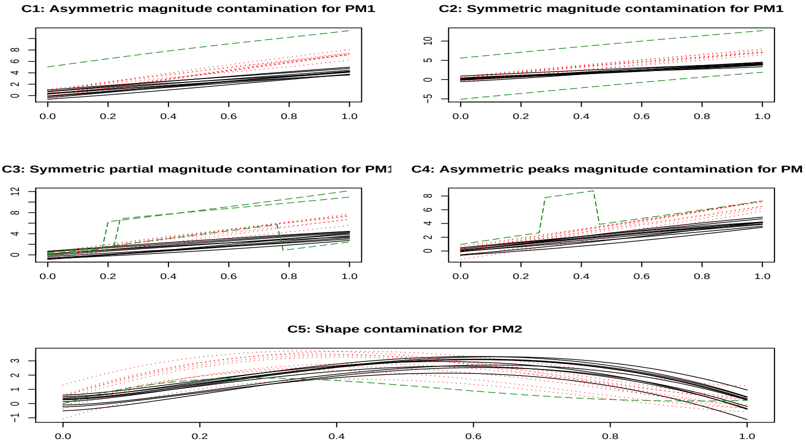
where $m_g(t)$ is a deterministic mean function characterizing the group g and $\epsilon(t)$ is a zero-mean Gaussian component.

- We consider two different scenarios for the **pair of mean functions**, $m_0(t)$ and $m_1(t)$:
  1. **PM1**: $m_0(t) = 4t; \quad m_1(t) = 7t.$
  2. **PM2**: $m_0(t) = 15(1-t)t^{1.5}; \quad m_1(t) = 15(1-t)^{1.5}t.$
- Through different covariance functions for $\epsilon(t)$, we consider two different **dependence structures** in the simulated data:
  1. **DS1**: $\mathbb{E}(\epsilon(t), \epsilon(s)) = 0.25 \exp\{-(t-s)^2\}, \quad t, s \in [0, 1].$
  2. **DS2**: $\mathbb{E}(\epsilon(t), \epsilon(s)) = 0.3 \exp\{-|t-s|/0.3\}, \quad t, s \in [0, 1].$

Simulation study: no contamination



Simulation study: with contamination



Simulation study: summary of the structure

- **No contamination**:
  1. 2 scenarios for the mean functions: PM1 and PM2.
  2. 2 scenarios for the covariances structures: DS1 and DS2.

4 scenarios

- **Contamination**:
  1. The contamination affects only $m_1(t)$ and its probability is $q = 0.1$.
  2. 5 scenarios for the contamination: C1, C2, C3, C4 and C5.
  3. 2 scenarios for the covariances structures: DS1 and DS2.

10 scenarios

- **Common features for all scenarios**:
  1. $R = 125$: number of replications for each scenario.
  2. $n_0 = n_1 = 50$: sample sizes for group 0 and 1.
  3. $n_{train} = n_{test} = 25$: training and test sample sizes for group 0 and 1.
  4. $m = 51$: number of equidistant points at which the curves are observed.

Simulation study: results in the no contamination scenarios

Linear means + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	1.66 *(1.16)*	1.81 *(1.07)*	1.66 *(1.01)*	1.79 *(1.11)*	1.74 *(1.10)*	1.63 *(1.12)*	1.78 *(1.07)*
WAD	1.41 *(1.17)*	1.65 *(1.11)*	1.58 *(1.07)*	1.52 *(1.13)*	1.50 *(1.10)*	1.58 *(1.13)*	1.38 *(1.11)*
WMD	10.50 *(1.17)*	2.18 *(1.11)*	11.04 *(1.07)*	3.92 *(1.13)*	8.66 *(1.10)*	3.12 *(1.13)*	1.58 *(1.11)*
k-NN				0.69 *(1.57)*			

Linear means + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	0.35 *(2.29)*	0.37 *(2.33)*	0.30 *(2.65)*	0.35 *(2.40)*	0.32 *(2.43)*	0.34 *(2.48)*	0.35 *(2.40)*
WAD	0.34 *(2.36)*	0.35 *(2.40)*	0.29 *(2.60)*	0.34 *(2.36)*	0.32 *(2.43)*	0.32 *(2.43)*	0.32 *(2.43)*
WMD	3.20 *(0.77)*	1.95 *(1.24)*	17.46 *(0.34)*	0.96 *(1.44)*	2.18 *(0.93)*	0.35 *(2.17)*	0.59 *(2.10)*
k-NN				0.30 *(2.51)*			

Nonlinear means + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.06 *(1.36)*	2.30 *(1.35)*	2.00 *(1.51)*	2.00 *(1.37)*	1.98 *(1.39)*	1.90 *(1.38)*	2.34 *(1.25)*
WAD	1.38 *(1.48)*	2.19 *(1.29)*	1.55 *(1.52)*	1.58 *(1.45)*	1.44 *(1.49)*	1.57 *(1.43)*	1.44 *(1.34)*
WMD	25.14 *(0.26)*	3.12 *(1.08)*	12.42 *(0.50)*	23.47 *(0.32)*	21.04 *(0.28)*	7.04 *(0.71)*	2.03 *(1.17)*
k-NN				0.88 *(1.57)*			

Nonlinear means + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	6.96 *(0.56)*	6.94 *(0.57)*	6.91 *(0.55)*	6.70 *(0.58)*	6.99 *(0.59)*	6.86 *(0.55)*	6.99 *(0.58)*
WAD	6.66 *(0.56)*	6.96 *(0.58)*	6.98 *(0.57)*	6.77 *(0.54)*	6.70 *(0.57)*	6.78 *(0.57)*	6.78 *(0.57)*
WMD	13.20 *(0.38)*	8.58 *(0.54)*	20.45 *(0.30)*	21.82 *(0.32)*	11.81 *(0.39)*	7.49 *(0.51)*	6.70 *(0.59)*
k-NN				6.59 *(0.58)*			

Simulation study: results in the no contamination scenarios

Linear means + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	1.66	1.81	1.66	1.79	1.74	1.63	1.78
	(1.16)	*(1.07)*	*(1.01)*	*(1.11)*	*(1.10)*	*(1.12)*	*(1.07)*
WAD	1.41	1.65	1.58	1.52	1.50	1.58	1.38
	(1.17)	*(1.11)*	*(1.07)*	*(1.13)*	*(1.10)*	*(1.13)*	*(1.11)*
WMD	10.50	2.18	11.04	3.92	8.66	3.12	1.58
	(1.17)	*(1.11)*	*(1.07)*	*(1.13)*	*(1.10)*	*(1.13)*	*(1.11)*
k-NN	0.69						
	(1.57)						

Linear means + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	0.35	0.37	0.30	0.35	0.32	0.34	0.35
	(2.29)	*(2.33)*	*(2.65)*	*(2.40)*	*(2.43)*	*(2.48)*	*(2.40)*
WAD	0.34	0.35	**0.29**	0.34	0.32	0.32	0.32
	(2.36)	*(2.40)*	*(2.60)*	*(2.36)*	*(2.43)*	*(2.43)*	*(2.43)*
WMD	3.20	1.95	17.46	0.96	2.18	0.35	0.59
	(0.77)	*(1.24)*	*(0.34)*	*(1.44)*	*(0.93)*	*(2.17)*	*(2.10)*
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Nonlinear means + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.06	2.30	2.00	2.00	1.98	1.90	2.34
	(1.36)	*(1.35)*	*(1.51)*	*(1.37)*	*(1.39)*	*(1.38)*	*(1.25)*
WAD	1.38	2.19	1.55	1.58	1.44	1.57	1.44
	(1.48)	*(1.29)*	*(1.52)*	*(1.45)*	*(1.49)*	*(1.43)*	*(1.34)*
WMD	25.14	3.12	12.42	23.47	21.04	7.04	2.03
	(0.26)	*(1.08)*	*(0.50)*	*(0.32)*	*(0.28)*	*(0.71)*	*(1.17)*
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Nonlinear means + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	6.96	6.94	6.91	6.70	6.99	6.86	6.99
	(0.56)	*(0.57)*	*(0.55)*	*(0.58)*	*(0.59)*	*(0.55)*	*(0.58)*
WAD	6.66	6.96	6.98	6.77	6.70	6.78	6.78
	(0.56)	*(0.58)*	*(0.57)*	*(0.54)*	*(0.57)*	*(0.57)*	*(0.57)*
WMD	13.20	8.58	20.45	21.82	11.81	7.49	6.70
	(0.38)	*(0.54)*	*(0.30)*	*(0.32)*	*(0.39)*	*(0.51)*	*(0.59)*
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Simulation study: results in the no contamination scenarios

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Simulation study: results in the no contamination scenarios

Linear means + Strong dependence

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Simulation study: symmetric contamination

Symmetric Total + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	4.51 *(0.65)*	4.50 *(0.64)*	4.35 *(0.63)*	4.48 *(0.61)*	4.40 *(0.65)*	4.43 *(0.64)*	4.51 *(0.65)*
WAD	4.88 *(0.59)*	4.22 *(0.63)*	4.19 *(0.63)*	4.32 *(0.65)*	4.38 *(0.64)*	4.30 *(0.64)*	5.26 *(0.62)*
WMD	14.53 *(0.43)*	4.19 *(0.86)*	11.74 *(0.44)*	7.25 *(0.69)*	12.69 *(0.47)*	6.53 *(0.78)*	2.56 *(1.19)*
k-NN				2.93 *(0.83)*			

Symmetric Total + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.54 *(0.86)*	2.59 *(0.86)*	2.72 *(0.85)*	2.59 *(0.86)*	2.58 *(0.86)*	2.58 *(0.87)*	2.61 *(0.86)*
WAD	3.34 *(0.75)*	2.59 *(0.87)*	2.62 *(0.84)*	2.77 *(0.80)*	2.77 *(0.79)*	2.66 *(0.85)*	3.66 *(0.75)*
WMD	6.75 *(0.66)*	3.84 *(0.90)*	16.77 *(0.42)*	5.34 *(1.02)*	6.27 *(0.84)*	4.26 *(1.33)*	1.33 *(1.71)*
k-NN				2.27 *(1.00)*			

Symmetric Partial + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	4.26 *(0.72)*	4.11 *(0.70)*	4.14 *(0.78)*	4.11 *(0.75)*	4.22 *(0.74)*	4.10 *(0.74)*	4.18 *(0.72)*
WAD	4.51 *(0.70)*	4.13 *(0.72)*	3.97 *(0.77)*	4.05 *(0.75)*	4.34 *(0.74)*	3.94 *(0.75)*	4.46 *(0.72)*
WMD	12.75 *(0.42)*	5.04 *(0.77)*	14.88 *(0.48)*	7.25 *(0.57)*	11.17 *(0.47)*	6.53 *(0.71)*	3.66 *(1.01)*
k-NN				3.17 *(0.82)*			

Symmetric Partial + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.70 *(0.90)*	2.70 *(0.89)*	2.69 *(0.90)*	2.67 *(0.90)*	2.70 *(0.90)*	2.69 *(0.90)*	2.67 *(0.89)*
WAD	3.31 *(0.85)*	2.66 *(0.91)*	2.69 *(0.91)*	2.72 *(0.89)*	2.88 *(0.84)*	2.66 *(0.92)*	3.23 *(0.84)*
WMD	5.17 *(0.74)*	3.94 *(0.85)*	19.42 *(0.47)*	4.94 *(0.78)*	4.29 *(0.89)*	3.31 *(1.36)*	1.65 *(1.41)*
k-NN				2.64 *(0.92)*			

Simulation study: symmetric contamination

Symmetric Total + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	4.51 *(0.65)*	4.50 *(0.64)*	4.35 *(0.63)*	4.48 *(0.61)*	4.40 *(0.65)*	4.43 *(0.64)*	4.51 *(0.65)*
WAD	4.88 *(0.59)*	4.22 *(0.63)*	4.19 *(0.63)*	4.32 *(0.65)*	4.38 *(0.64)*	4.30 *(0.64)*	5.26 *(0.62)*
WMD	14.53 *(0.43)*	4.19 *(0.86)*	11.74 *(0.44)*	7.25 *(0.69)*	12.69 *(0.47)*	6.53 *(0.78)*	**2.56** *(1.19)*
k-NN				**2.93** *(0.83)*			

Symmetric Total + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.54 *(0.86)*	2.59 *(0.86)*	2.72 *(0.85)*	2.59 *(0.86)*	2.58 *(0.86)*	2.58 *(0.87)*	2.61 *(0.86)*
WAD	3.34 *(0.75)*	2.59 *(0.87)*	2.62 *(0.84)*	2.77 *(0.80)*	2.77 *(0.79)*	2.66 *(0.85)*	3.66 *(0.75)*
WMD	6.75 *(0.66)*	3.84 *(0.90)*	16.77 *(0.42)*	5.34 *(1.02)*	6.27 *(0.84)*	4.26 *(1.33)*	**1.33** *(1.71)*
k-NN				**2.27** *(1.00)*			

Symmetric Partial + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	4.26 *(0.72)*	4.11 *(0.70)*	4.14 *(0.78)*	4.11 *(0.75)*	4.22 *(0.74)*	4.10 *(0.74)*	4.18 *(0.72)*
WAD	4.51 *(0.70)*	4.13 *(0.72)*	3.97 *(0.77)*	4.05 *(0.75)*	4.34 *(0.74)*	3.94 *(0.75)*	4.46 *(0.72)*
WMD	12.75 *(0.42)*	5.04 *(0.77)*	14.88 *(0.48)*	7.25 *(0.57)*	11.17 *(0.47)*	6.53 *(0.71)*	**3.66** *(1.01)*
k-NN				**3.17** *(0.82)*			

Symmetric Partial + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.70 *(0.90)*	2.70 *(0.89)*	2.69 *(0.90)*	2.67 *(0.90)*	2.70 *(0.90)*	2.69 *(0.90)*	2.67 *(0.89)*
WAD	3.31 *(0.85)*	2.66 *(0.91)*	2.69 *(0.91)*	2.72 *(0.89)*	2.88 *(0.84)*	2.66 *(0.92)*	3.23 *(0.84)*
WMD	5.17 *(0.74)*	3.94 *(0.85)*	19.42 *(0.47)*	4.94 *(0.78)*	4.29 *(0.89)*	3.31 *(1.36)*	**1.65** *(1.41)*
k-NN				**2.64** *(0.92)*			

Simulation study: shape contamination

Shape + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.59 *(1.37)*	2.56 *(1.23)*	2.29 *(1.15)*	2.19 *(1.23)*	2.30 *(1.28)*	2.18 *(1.16)*	2.64 *(1.28)*
WAD	1.94 *(1.32)*	2.02 *(1.30)*	2.02 *(1.21)*	1.89 *(1.28)*	1.84 *(1.32)*	1.76 *(1.34)*	2.02 *(1.26)*
WMD	24.40 *(0.26)*	3.39 *(0.92)*	12.99 *(0.39)*	23.57 *(0.29)*	20.59 *(0.31)*	6.75 *(0.63)*	2.19 *(1.17)*
k-NN				0.99 *(1.69)*			

Shape + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	6.42 *(0.52)*	6.35 *(0.53)*	6.27 *(0.54)*	6.18 *(0.50)*	6.34 *(0.52)*	6.19 *(0.54)*	6.24 *(0.52)*
WAD	6.16 *(0.53)*	6.16 *(0.52)*	6.40 *(0.53)*	6.10 *(0.54)*	6.08 *(0.54)*	6.08 *(0.54)*	6.00 *(0.55)*
WMD	14.27 *(0.35)*	9.39 *(0.52)*	21.10 *(0.28)*	23.02 *(0.29)*	12.70 *(0.38)*	7.02 *(0.59)*	7.01 *(0.58)*
k-NN				6.13 *(0.53)*			

Simulation study: shape contamination

Shape + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	2.59 *(1.37)*	2.56 *(1.23)*	2.29 *(1.15)*	2.19 *(1.23)*	2.30 *(1.28)*	2.18 *(1.16)*	2.64 *(1.28)*
WAD	1.94 *(1.32)*	2.02 *(1.30)*	2.02 *(1.21)*	1.89 *(1.28)*	1.84 *(1.32)*	**1.76** *(1.34)*	2.02 *(1.26)*
WMD	24.40 *(0.26)*	3.39 *(0.92)*	12.99 *(0.39)*	23.57 *(0.29)*	20.59 *(0.31)*	6.75 *(0.63)*	2.19 *(1.17)*
k-NN				**0.99** *(1.69)*			

Shape + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	6.42 *(0.52)*	6.35 *(0.53)*	6.27 *(0.54)*	6.18 *(0.50)*	6.34 *(0.52)*	6.19 *(0.54)*	6.24 *(0.52)*
WAD	6.16 *(0.53)*	6.16 *(0.52)*	6.40 *(0.53)*	6.10 *(0.54)*	6.08 *(0.54)*	6.08 *(0.54)*	**6.00** *(0.55)*
WMD	14.27 *(0.35)*	9.39 *(0.52)*	21.10 *(0.28)*	23.02 *(0.29)*	12.70 *(0.38)*	7.02 *(0.59)*	7.01 *(0.58)*
k-NN				**6.13** *(0.53)*			

Simulation study: asymmetric contamination

Asymmetric Total + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	1.73	1.58	2.54	2.02	2.05	2.06	1.79
	(1.56)	*(1.67)*	*(1.24)*	*(1.46)*	*(1.44)*	*(1.42)*	*(1.84)*
WAD	2.34	1.34	1.70	1.87	1.95	1.62	2.66
	(1.37)	*(1.44)*	*(1.24)*	*(1.29)*	*(1.30)*	*(1.28)*	*(1.32)*
WMD	10.62	3.18	9.84	3.89	8.90	3.12	2.42
	(0.44)	*(1.01)*	*(0.53)*	*(0.79)*	*(0.50)*	*(0.90)*	*(1.14)*
k-NN				0.45 *(2.26)*			

Asymmetric Total + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	0.46	0.48	1.68	0.61	0.54	0.51	0.56
	(1.91)	*(2.01)*	*(1.57)*	*(2.06)*	*(1.95)*	*(2.04)*	*(2.01)*
WAD	2.06	0.46	0.77	1.10	1.14	0.82	2.54
	(1.38)	*(1.91)*	*(1.80)*	*(1.60)*	*(1.59)*	*(1.76)*	*(1.35)*
WMD	4.22	3.10	13.57	1.58	3.34	0.75	1.31
	(0.70)	*(1.00)*	*(0.43)*	*(1.04)*	*(0.80)*	*(1.50)*	*(1.54)*
k-NN				0.40 *(2.20)*			

Asymmetric Peaks + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	1.86	1.73	1.78	1.78	1.90	1.66	1.82
	(1.00)	*(1.07)*	*(0.96)*	*(0.99)*	*(0.93)*	*(1.09)*	*(0.99)*
WAD	1.76	1.57	1.70	1.79	1.87	1.57	1.73
	(0.99)	*(1.13)*	*(1.03)*	*(0.96)*	*(0.95)*	*(1.09)*	*(0.99)*
WMD	10.21	4.18	12.08	3.95	8.40	3.39	3.47
	(0.46)	*(0.89)*	*(0.45)*	*(0.71)*	*(0.52)*	*(0.78)*	*(0.97)*
k-NN				0.69 *(1.85)*			

Asymmetric Peaks + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	0.34	0.38	0.34	0.45	0.38	0.37	0.42
	(2.48)	*(2.36)*	*(2.36)*	*(2.03)*	*(2.26)*	*(2.43)*	*(2.23)*
WAD	0.50	0.40	0.34	0.43	0.51	0.38	0.50
	(1.82)	*(2.29)*	*(2.36)*	*(2.00)*	*(1.85)*	*(2.26)*	*(1.89)*
WMD	3.28	3.63	15.47	1.02	2.27	0.72	1.73
	(0.84)	*(1.00)*	*(0.39)*	*(1.58)*	*(1.04)*	*(1.59)*	*(1.59)*
k-NN				0.29 *(2.60)*			

Simulation study: asymmetric contamination

Asymmetric Total + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	1.73 *(1.56)*	1.58 *(1.67)*	2.54 *(1.24)*	2.02 *(1.46)*	2.05 *(1.44)*	2.06 *(1.42)*	1.79 *(1.84)*
WAD	2.34 *(1.37)*	**1.34** *(1.44)*	1.70 *(1.24)*	1.87 *(1.29)*	1.95 *(1.30)*	1.62 *(1.28)*	2.66 *(1.32)*
WMD	10.62 *(0.44)*	3.18 *(1.01)*	9.84 *(0.53)*	3.89 *(0.79)*	8.90 *(0.50)*	3.12 *(0.90)*	2.42 *(1.14)*
k-NN				**0.45** *(2.26)*			

Asymmetric Total + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	**0.46** *(1.91)*	0.48 *(2.01)*	1.68 *(1.57)*	0.61 *(2.06)*	0.54 *(1.95)*	0.51 *(2.04)*	0.56 *(2.01)*
WAD	2.06 *(1.38)*	**0.46** *(1.91)*	0.77 *(1.80)*	1.10 *(1.60)*	1.14 *(1.59)*	0.82 *(1.76)*	2.54 *(1.35)*
WMD	4.22 *(0.70)*	3.10 *(1.00)*	13.57 *(0.43)*	1.58 *(1.04)*	3.34 *(0.80)*	0.75 *(1.50)*	1.31 *(1.54)*
k-NN				**0.40** *(2.20)*			

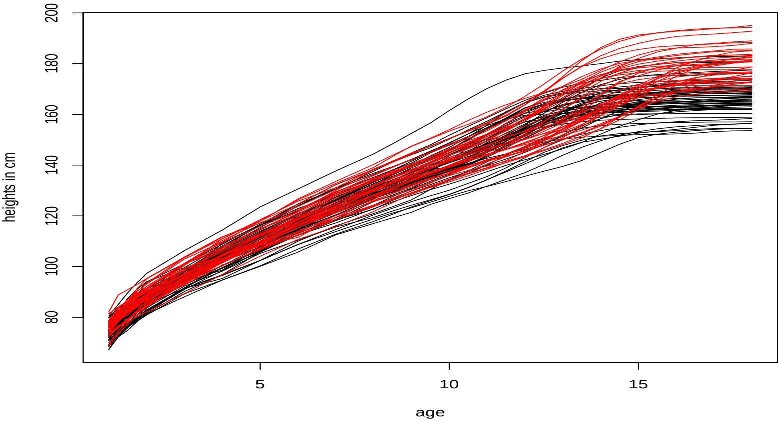
Asymmetric Peaks + Strong dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	1.86 *(1.00)*	1.73 *(1.07)*	1.78 *(0.96)*	1.78 *(0.99)*	1.90 *(0.93)*	1.66 *(1.09)*	1.82 *(0.99)*
WAD	1.76 *(0.99)*	**1.57** *(1.13)*	1.70 *(1.03)*	1.79 *(0.96)*	1.87 *(0.95)*	**1.57** *(1.09)*	1.73 *(0.99)*
WMD	10.21 *(0.46)*	4.18 *(0.89)*	12.08 *(0.45)*	3.95 *(0.71)*	8.40 *(0.52)*	3.39 *(0.78)*	3.47 *(0.97)*
k-NN				**0.69** *(1.85)*			

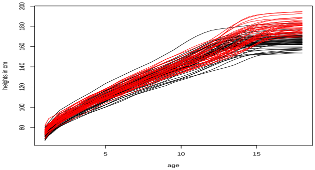
Asymmetric Peaks + Weak dependence

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	**0.34** *(2.48)*	0.38 *(2.36)*	**0.34** *(2.36)*	0.45 *(2.03)*	0.38 *(2.26)*	0.37 *(2.43)*	0.42 *(2.23)*
WAD	0.50 *(1.82)*	0.40 *(2.29)*	**0.34** *(2.36)*	0.43 *(2.00)*	0.51 *(1.85)*	0.38 *(2.26)*	0.50 *(1.89)*
WMD	3.28 *(0.84)*	3.63 *(1.00)*	15.47 *(0.39)*	1.02 *(1.58)*	2.27 *(1.04)*	0.72 *(1.59)*	1.73 *(1.59)*
k-NN				**0.29** *(2.60)*			

Growth data

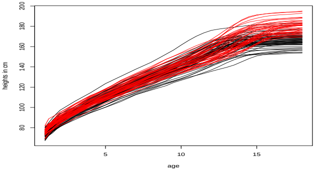


Growth data study



- The graph shows 93 **growth curves**: 54 heights of girls (black) and 39 heights of boys (red).
- The curves are observed at a common set of 31 nonequidistant ages between 1 and 18 years.
- We increase the number of evaluation points via linear interpolation and we obtain growth curves that are evaluated at a common discretized set of 69 equidistant ages. Clearly, other techniques can be used for this task.

Growth data study



- The growth data have a **strong dependence structure**.
- From our point of view, these data are interesting because we can not discard the presence of some **outlying curve**, and in particular among the heights of the girls. Moreover, it is the real dataset used by López-Pintado and Romo (2006) and Cuevas, Febrero and Fraiman (2007) to show their depth-based functional supervised classification methods.
- **Features of the study**: $R = 140$; $n_{train_g} = 40$, $n_{train_b} = 30$, $n_{test_g} = 14$, $n_{test_b} = 9$.

Growth data study: results

Means and coefficients of variation for the misclassification percentages for the growth curves

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	14.81 *(0.60)*	10.06 *(0.85)*	20.00 *(0.47)*	19.41 *(0.51)*	17.11 *(0.54)*	19.13 *(0.52)*	12.55 *(0.72)*
WAD	13.88 *(0.59)*	8.76 *(0.89)*	15.34 *(0.55)*	14.97 *(0.56)*	14.35 *(0.57)*	14.88 *(0.56)*	13.17 *(0.61)*
WMD	29.57 *(0.39)*	5.16 *(0.88)*	14.10 *(0.49)*	31.18 *(0.33)*	26.18 *(0.43)*	17.76 *(0.47)*	3.39 *(1.01)*
k-NN				3.88 *(0.79)*			

Growth data study: results

Means and coefficients of variation for the misclassification percentages for the growth curves

Method	FMD	HMD	RTD	IDD	MBD	**FSD**	**KFSD**
DTM	14.81 *(0.60)*	10.06 *(0.85)*	20.00 *(0.47)*	19.41 *(0.51)*	17.11 *(0.54)*	19.13 *(0.52)*	12.55 *(0.72)*
WAD	13.88 *(0.59)*	**8.76** *(0.89)*	15.34 *(0.55)*	14.97 *(0.56)*	14.35 *(0.57)*	14.88 *(0.56)*	13.17 *(0.61)*
WMD	29.57 *(0.39)*	**5.16** *(0.88)*	14.10 *(0.49)*	31.18 *(0.33)*	26.18 *(0.43)*	17.76 *(0.47)*	**3.39** *(1.01)*
k-NN				**3.88** *(0.79)*			

Conclusions

- We have introduced **two new functional depths**: the functional spatial depth and the kernelized functional spatial depth.
- The main novelty introduced by **FSD** consists in the connection between its definition and the notion of functional spatial quantiles.
- With **KFSD** we have addressed the study of functional datasets that require analyses at a local level.
- FSD and KFSD have been used to solve **supervised functional classification problems**, especially in situations where the functional samples were contaminated, but we have also considered noncontaminated scenarios.

Conclusions

- **KFSD** has proved to be the best depth for the considered depth-based methods, and a good competitor for k-NN. In particular, we would like to highlight its performances under the following situations:
  1. Functional data with **weak dependence structure**.
  2. **Contaminated functional datasets**, in particular symmetric and shape contamination.
  3. A **real data case** characterized by a strong dependence structure and the potential presence of outlying curves.
- **What next?**
  1. Depth-based outlier detection methods. If possible, new depth-based supervised classification methods and depth-based clustering methods.
  2. Data-driven procedures for the choices about the kernel and the bandwidth for KFSD.
  3. Theoretical study of FSD based on its connection with the notion of functional spatial quantiles.

Conclusions

- This presentation is based on Sguera, Galeano and Lillo (2012) available at

  `http://hdl.handle.net/10016/14331`

THANK YOU FOR YOUR ATTENTION.